SOME STATISTICAL PROPERTIES OF SELECTED INVENTORY MODELS

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SOME STATISTICAL PROPERTIES OF SELECTED INVENTORY MODELS

Murray A. Geisler*

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In the study of inventory policies, ~~one~~ is interested not only in the mean values of such important random variables as number of shortages per time period, but also in their variance and covariance properties. Such additional properties are of interest in interpreting the stability of an expected value, under assumed inventory policies and parameters, and in using stochastic or Monte Carlo models to calculate estimates of the expected values by sampling techniques. In this paper, ~~we examine~~ comparatively simple inventory models, and ~~derive~~ the expected value, variance, and selected covariance and correlations of the random variables representing stock on hand, shortages per period, overages per period and reorder quantity, ~~each~~ of which will be defined below.

I. INVENTORY MODEL WITH ZERO PROCUREMENT LEAD TIME

First, we consider an inventory model with zero procurement lead time which is governed by (S, s) policies. We assume that a particular set of values (S, s) has been selected, so that whenever the stock level x falls below s, then positive ordering is immediately enacted to raise the level to S with immediate delivery. When the quantity of goods in supply x exceeds s, then no ordering is done. We allow x to assume any possible

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[redacted]

real value. A negative stock level should be interpreted as the amount owed to consumption. Thus, all demand will be ultimately satisfied, and therefore it is meaningful to refer to negative stock levels. We also assume that the density of demand $f(\xi)$ is known, so that in each time period, a demand ξ has probability $f(\xi)$ of occurring. Then, if x_n = stock level at end of period n, we have:

$$x_{n+1} = \begin{cases} x_n - \xi & \text{if} \quad s < x_n \leq S \\ S - \xi & \text{if} \quad x_n \leq s \end{cases}$$

1. STATISTICAL PROPERTIES OF THE STOCK LEVEL x_n

We first solve for $\mathrm{Cov}\{x_n\ x_{n+1}\}$, which is given by:

$$\mathrm{Cov}\{x_n\ x_{n+1}\} = E\{x_n\ x_{n+1}\} - E\{x_n\}\ E\{x_{n+1}\}$$

We can then define $E\{x_n\ x_{n+1}\}$ as follows:

$$E\{x_n\ x_{n+1}\} = \int E\{x_{n+1}\ x_n | x_n\}\ \phi(x_n)dx_n$$

$$= \int x_n E\{x_{n+1} | x_n\}\ \phi(x_n)dx_n$$

Using the above transition law, we get:

$$E\{x_{n+1} | x_n\} = \begin{cases} x_n - m & \text{if} \quad s < x_n \leq S \\ S - m & \text{if} \quad x_n \leq s \end{cases}$$

Where $m = E(\xi)$.

Therefore,

$$E\{x_n \; x_{n+1}\} = (S - m)\int_{-\infty}^{s} x_n \phi(x_n)dx_n + \int_{s}^{S} x_n(x_n - m)\phi(x_n)dx_n$$

Now, we solve this expression for the special case of exponential demand: $f(\xi) = \lambda e^{-\lambda\xi}$. For this continuous density function, the stationary distribution of x is given by.*

$$\phi(x) = \begin{cases} \dfrac{\lambda}{1 + \lambda\Delta}\,; & \text{if} \quad s < x \leq S \\[2ex] \dfrac{\lambda e^{-\lambda(s - x)}}{1 + \lambda\Delta}\,; & \text{if} \quad x \leq s \end{cases}$$

where $\Delta = S - s$.

Substituting $\phi(x)$ for $\phi(x_n)$ in the above expression for $E\{x_n \; x_{n+1}\}$ we obtain:

$$E\{x_n \; x_{n+1}\} = \frac{(S - \frac{1}{\lambda})}{1 + \lambda\Delta}\int_{-\infty}^{s} \lambda x e^{-\lambda(s - x)}dx + \frac{\lambda}{1 + \lambda\Delta}\int_{s}^{S} x(x - \tfrac{1}{\lambda})dx$$

where $m = \frac{1}{\lambda}$.

The first term on the right can then be integrated, and for convenience, we let $y = s - x$. We then get:

$$\frac{S - \frac{1}{\lambda}}{1 + \lambda\Delta}\int_{-\infty}^{s} \lambda x e^{-\lambda(s - x)}dx = \frac{S - \frac{1}{\lambda}}{1 + \lambda\Delta}\int_{0}^{\infty} \lambda(s - y)e^{-\lambda y}dy = \frac{(S - \frac{1}{\lambda})}{1 + \lambda\Delta}\left\{s - \frac{1}{\lambda}\right\}$$

*Arrow, K., Karlin, S., and Scarf, H., Studies in the Mathematical Theory of Inventory and Production, 1959, Chapter 14.

Also,

$$\frac{\lambda}{1+\lambda\Delta}\int_s^S x(x-\frac{1}{\lambda})dx = \frac{\lambda}{1+\lambda\Delta}\left(\frac{S^3}{3}-\frac{s^3}{3}-\frac{S^2}{2\lambda}+\frac{s^2}{2\lambda}\right)$$

Combining, we get

$$E\{x_n\ x_{n+1}\} = \frac{Ss-\frac{S}{\lambda}-\frac{s}{\lambda}+\frac{1}{\lambda^2}+\frac{\lambda S^3}{3}-\frac{\lambda s^3}{3}-\frac{S^2}{2}+\frac{s^2}{2}}{1+\lambda\Delta}$$

Now, we solve for $E\{x_n\}$

$$E\{x_n\} = \int E\{x_{n+1}\,|\,x_n\}\phi(x_n)dx_n$$

$$= \int_{-\infty}^{s}(S-\frac{1}{\lambda})\frac{\lambda e^{-\lambda(s-x)}}{1+\lambda\Delta}dx + \int_s^S(x-\frac{1}{\lambda})\frac{\lambda}{1+\lambda\Delta}dx$$

$$= \frac{S-\frac{1}{\lambda}}{1+\lambda\Delta}+\frac{\lambda}{1+\lambda\Delta}\left(\frac{S^2}{2}-\frac{s^2}{2}-\frac{S}{\lambda}+\frac{s}{\lambda}\right)$$

$$= \frac{s-\frac{1}{\lambda}+\lambda\frac{S^2}{2}-\lambda\frac{s^2}{2}}{1+\lambda\Delta}$$

Therefore:

$$[E\{x_n\}]^2 = \frac{s^2+\frac{1}{\lambda^2}+\frac{\lambda^2S^4}{4}+\frac{\lambda^2s^4}{4}-\frac{2s}{\lambda}+\lambda sS^2-\lambda s^3-S^2+s^2-\frac{\lambda^2s^2S^2}{2}}{(1+\lambda\Delta)^2}$$

We can now find the $\text{Cov}\{x_n\ x_{n+1}\}$ by:

$$\text{Cov}\{x_n\ x_{n+1}\} = E\{x_n\ x_{n+1}\} - [E\{x_n\}]^2$$

since $E\{x_n\} = E\{x_{n+1}\}$

Therefore:

$$\operatorname{Cov}\{x_n\ x_{n+1}\} = \frac{\frac{\lambda S^3}{3} - \frac{\lambda s^3}{3} - \frac{S^2}{2} + \frac{s^2}{2} + Ss - \frac{S}{\lambda} - \frac{s}{\lambda} + \frac{1}{\lambda^2}}{1 + \lambda\Delta}$$

$$- \frac{s^2 + \frac{1}{\lambda^2} + \frac{\lambda^2 S^4}{4} + \frac{\lambda^2 s^4}{4} - \frac{2s}{\lambda} + \lambda s S^2 - \lambda s^3 - S^2 + s^2 - \frac{\lambda^2 S^2 s^2}{2}}{(1 + \lambda\Delta)^2}$$

Simplifying, we get

$$\operatorname{Cov}\{x_n\ x_{n+1}\} = \frac{\frac{(S - s)^2}{12}\,[\lambda^2(S - s)^2 - 2\lambda(S - s) - 6]}{[1 + \lambda(S - s)]^2}$$

We next derive $\operatorname{Var}\{x_n\}$, which is given by

$$\operatorname{Var}\{x_n\} = E\{x_n^2\} - [E\{x_n\}]^2$$

We know that:

$$E\{x_n^2\} = \int E\{x_n^2 | x_n\}\,\phi(x_n)dx_n = \int x_n^2\phi(x_n)dx_n$$

Solving this expression for the case of exponential demand: $f(\xi) = \lambda e^{-\lambda\xi}$,

we get

$$E\{x_n^2\} = \int x^2\,\phi(x)dx = \int_s^S \frac{x^2\lambda}{1 + \lambda\Delta}\,dx + \int_{-\infty}^{s} \frac{\lambda x^2 e^{-\lambda(s - x)}}{1 + \lambda\Delta}\,dx$$

$$= \frac{\lambda}{1+\lambda\Delta}\left(\frac{S^3}{3} - \frac{s^3}{3}\right) + \frac{1}{1+\lambda\Delta}\int_0^\infty \lambda(s^2 - 2sy + y^2)e^{-\lambda y}dy$$

substituting in the second expression $y = s - x$. We then get

$$E\{x_n^2\} = \frac{\lambda}{1+\lambda\Delta}\left(\frac{S^3}{3} - \frac{s^3}{3}\right) + \frac{1}{1+\lambda\Delta}\left(s^2 - \frac{2s}{\lambda} + \frac{2}{\lambda^2}\right)$$

To obtain Var$\{x_n\}$, we must now subtract $[E\{x_n\}]^2$ from $E\{x_n^2\}$. We then get

$$\mathrm{Var}\{x_n\} = \frac{\lambda\left(\frac{S^3}{3} - \frac{s^3}{3}\right) + s^2 - \frac{2s}{\lambda} + \frac{2}{\lambda^2}}{1+\lambda\Delta} - \left(\frac{s - \frac{\lambda s^2}{2} + \frac{\lambda S^2}{2} + \frac{1}{\lambda}}{1+\lambda\Delta}\right)^2$$

Reducing this expression we get

$$\mathrm{Var}\{x_n\} = \frac{\frac{\lambda^2}{12}(S-s)^4 + \frac{\lambda}{3}(S-s)^3 + (S-s)^2 + \frac{2}{\lambda}(S-s) + \frac{1}{\lambda^2}}{(1+\lambda\Delta)^2}$$

Knowing Cov$\{x_n\, x_{n+1}\}$ and Var$\{x_n\}$, we can also find the correlation between x_n and x_{n+1}. This is given by:

$$\rho_{x_n\, x_{n+1}} = \frac{\mathrm{Cov}\{x_n\, x_{n+1}\}}{\mathrm{Var}\{x_n\}}.$$

Using the results obtained for the case of exponential demand, we find that the correlation between x_n and x_{n+1} is given by

$$\rho_{x_n\, x_{n+1}} = \frac{\frac{(S-s)^2}{12}\left[\lambda^2(S-s)^2 - 2\lambda(S-s) - 6\right]}{\frac{\lambda^2}{12}(S-s)^4 + \frac{\lambda}{3}(S-s)^3 + (S-s)^2 + \frac{2}{\lambda}(S-s) + \frac{1}{\lambda^2}}$$

The following table of $\rho_{x_n \, x_{n+1}}$ has been computed for a series of values of λ and $S - s$ giving the following results:

Table of $\rho_{x_n \, x_{n+1}}$

S - s	λ: .01	.1	1.0	10	100
1	-.000	-.005	-.13	.48	.95
5	-.001	-.06	.14	.85	.96
10	-.006	-.14	.48	.96	.99
25	-.03	-.30	.77	.99	.99+
50	-.10	.16	.91	.99+	.99+
100	-.14	.49	.96	.99+	.99+

Further, the sequence x_n is a regular and stationary Markov process, and from the properties of stationary Markov processes, we know that*

$$R(p) = a^p R(0)$$

where $R(p) = \mathrm{Cov}\left\{x_n \, x_{n+p}\right\}$ and $R(0) = \mathrm{Var}\left\{x_n\right\}$. Therefore

$$R(1) = aR(0) \; ,$$

or

$$\mathrm{Cov}\left\{x_n \, x_{n+1}\right\} = a \, \mathrm{Var}\left\{x_n\right\}$$

*Doob, J. L., Stochastic Processes, Chapt. 10.

so that

$$a = \frac{\mathrm{Cov}\{x_n\ x_{n+1}\}}{\mathrm{Var}\{x_n\}} = \rho_{x_n\ x_{n+1}}$$

Consequently

$$\mathrm{Cov}\{x_n\ x_{n+p}\} = \rho^p_{x_n\ x_{n+1}}\ \mathrm{Var}\{x_n\}$$

so that

$$\rho_{x_n\ x_{n+p}} = \frac{\mathrm{Cov}\{x_n\ x_{n+p}\}}{\mathrm{Var}\{x_n\}} = \rho^p_{x_n\ x_{n+1}}$$

Thus, the entire correlation function between x_n and x_{n+p} for all p can be obtained from knowledge of $\rho_{x_n\ x_{n+1}}$. Since we have $\rho_{x_n\ x_{n+1}}$ for the exponential distribution, we therefore can compute the correlations $\rho_{x_n\ x_{n+1}}$ for all p, since the inventory model we are studying is a stationary Markov process.

2. STATISTICAL PROPERTIES OF THE SHORTAGES y_n

We assume the same inventory model, as described above, with zero procurement lead time. Then, if y_n = shortages in n-th period, we have:

$$y_n = \begin{cases} 0 & \text{if} \quad x_n \geq 0 \\ -x_n & \text{if} \quad x_n < 0 \end{cases}$$

We first seek $\mathrm{Cov}\{y_n\ y_{n+1}\} = E\{y_n\ y_{n+1}\} - E\{y_n\}E\{y_{n+1}\}$.

Thus:

$$E(y_n\, y_{n+1}) = \int_{-\infty}^{\infty} E(y_n\, y_{n+1} | x_n)\phi(x_n)dx_n$$

$$E(y_n\, y_{n+1}) = \int_{-\infty}^{0} x_n E(y_{n+1} | x_n)\phi(x_n)dx_n$$

$$(y_{n+1} | x_n < 0) = \begin{cases} 0, & \text{if } \xi \leq S \\ \xi - S, & \text{if } \xi > S \end{cases}$$

$$E(y_{n+1} | x_n < 0) = \int_S^{\infty} (\xi - S)f(\xi)d\xi$$

$$= \int_S^{\infty} \xi f(\xi)d\xi - S\int_S^{\infty} f(\xi)d\xi$$

We now let $f(\xi) = \lambda e^{-\lambda\xi}$, the exponential distribution. Then:

$$E(y_{n+1} | x_n < 0) = \lambda\int_S^{\infty} \xi e^{-\lambda\xi}d\xi - S\lambda\int_S^{\infty} e^{-\lambda\xi}d\xi$$

$$= Se^{-\lambda S} + \frac{e^{-\lambda S}}{\lambda} - Se^{-\lambda S} = \frac{e^{-\lambda S}}{\lambda}$$

We then have:

$$\phi(x_n)dx_n = \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta}$$

$$\text{if } x_n < s \quad \text{for} \quad f(\xi) = \lambda e^{-\lambda\xi}$$

$$E(y_n\, y_{n+1}) = -\frac{e^{-\lambda S}e^{-\lambda s}}{1 + \lambda\Delta}\int_{-\infty}^{0} x_n e^{\lambda x_n}dx_n$$

$$= -\frac{e^{-\lambda(S+s)}}{1+\lambda\Delta}\left[-\frac{1}{\lambda^2}\right] = \frac{e^{-\lambda(S+s)}}{\lambda^2(1+\lambda\Delta)}$$

Solving for $E(y_n)$:

$$E(y_n) = E(y_{n+1}) = \int_{-\infty}^{\infty} E(y_n|x_n)\,\phi(x_n)dx_n$$

$$= -\int_{-\infty}^{0} x_n\phi(x_n)dx_n$$

$$E(y_n) = -\frac{e^{-\lambda s}}{1+\lambda\Delta}\int_{-\infty}^{0}\lambda x_n e^{\lambda x_n}dx_n$$

$$= \frac{e^{-\lambda s}}{\lambda(1+\lambda\Delta)}$$

Therefore:

$$Cov(y_n\ y_{n+1}) = \frac{e^{-\lambda(S+s)}}{\lambda^2(1+\lambda\Delta)} - \frac{e^{-2\lambda s}}{\lambda^2(1+\lambda\Delta)^2}$$

$$= \frac{e^{-\lambda s}}{\lambda^2(1+\lambda\Delta)}\left(e^{-\lambda S} - \frac{e^{-\lambda s}}{1+\lambda\Delta}\right)$$

We further note that:

$$Cov(y_n\ y_{n+1}) = \frac{e^{-2\lambda s}}{\lambda^2(1+\lambda\Delta)^2}\left(\frac{1+\lambda\Delta}{e^{\lambda\Delta}} - 1\right)$$

But

$$\frac{1+\lambda\Delta}{e^{\lambda\Delta}} \leq 1, \text{ so that } \quad Cov\ (y_n\ y_{n+1}) \leq 0$$

We now seek $\mathrm{Var}(y_n)$ to complete the correlation.

$$\mathrm{Var}(y_n) = E(y_n^2) - (Ey_n)^2$$

$$E(y_n^2) = \int_{-\infty}^{\infty} E(y_n^2 | x_n)\, \phi(x_n) dx_n$$

$$= \int_{-\infty}^{0} x_n^2\, \phi(x_n) dx_n$$

$$E(y_n^2) = \frac{e^{-\lambda s}}{1 + \lambda\Delta} \int_{-\infty}^{0} \lambda x_n^2 e^{\lambda x_n} dx_n$$

$$= \frac{2e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)}$$

Substituting:

$$\mathrm{Var}(y_n) = \frac{2e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)} - \frac{e^{-2\lambda s}}{\lambda^2(1 + \lambda\Delta)^2}$$

$$= \frac{e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)}\left(2 - \frac{e^{-\lambda s}}{1 + \lambda\Delta}\right)$$

Therefore, the correlation $\rho_{y_n\, y_{n+1}}$ is given by:

$$\rho_{y_n\, y_{n+1}} = \frac{\mathrm{Cov}(y_n\, y_{n+1})}{\mathrm{Var}\,(y_n)} = \frac{\dfrac{e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)}\left(e^{-\lambda S} - \dfrac{e^{-\lambda s}}{1 + \lambda\Delta}\right)}{\dfrac{e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)}\left(2 - \dfrac{e^{-\lambda s}}{1 + \lambda\Delta}\right)}$$

$$= \frac{e^{-\lambda S} - \dfrac{e^{-\lambda s}}{1 + \lambda\Delta}}{2 - \dfrac{e^{-\lambda s}}{1 + \lambda\Delta}} = \frac{e^{-\lambda S}(1 + \lambda\Delta) - e^{-\lambda s}}{2(1 + \lambda\Delta) - e^{-\lambda s}}$$

From the above result that Cov $(y_n\, y_{n+1}) \leq 0$, we also note that

$\rho_{y_n\, y_{n+1}} \leq 0$. Further, since y_n is not a Markov process, we cannot infer the behavior of $\rho_{y_n\, y_{n+p}}$, $p = 2, 3 \ldots$ from $\rho_{y_n\, y_{n+1}}$.

3. STATISTICAL PROPERTIES OF THE OVERAGE v_n

We still assume the same inventory model, as above, with zero procurement lead time. By 'overage', we mean the positive amount of stock left at the end of the period before ordering. If v_n = overage in n-th period, then

$$v_n = \begin{cases} x_n, & \text{if} \quad x_n > 0 \\ 0, & \text{if} \quad x_n \leq 0 \end{cases}$$

Recapitulating:

$$\phi(x_n) = \begin{cases} \dfrac{\lambda}{1 + \lambda\Delta}\,; & \text{if} \quad s < x_n \leq S \\ \dfrac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta}\,; & \text{if} \quad x_n \leq s \end{cases}$$

$$x_{n+1} = \begin{cases} x_n - \xi\,; & \text{if} \quad x_n > s \\ S - \xi\,; & \text{if} \quad x_n \leq s \end{cases}$$

We now seek Cov $(v_n\, v_{n+1})$, where:

$$\text{Cov}\,(v_n\, v_{n+1}) = E(v_n\, v_{n+1}) - E(v_n)\, E(v_{n+1})$$

We have that:

$$E(v_n\, v_{n+1}) = \int_{-\infty}^{\infty} E(v_n\, v_{n+1} | x_n)\, \phi\,(x_n) dx_n$$

$$= \int_0^{\infty} x_n E(v_{n+1} | x_n)\, \phi\,(x_n) dx_n$$

$$(v_{n+1} | 0 < x_n \leq s) = \begin{cases} 0\,, & \text{if } \xi \geq S \\ S - \xi\,, & \text{if } \xi < S \end{cases}$$

$$(v_{n+1} | s < x_n \leq S) = \begin{cases} 0\,, & \text{if } \xi \geq x_n \\ x_n - \xi\,, & \text{if } \xi < x_n \end{cases}$$

Thus:

$$E(v_{n+1} | 0 < x_n \leq s) = \int_0^S (S - \xi) f(\xi) d\xi = \int_0^S (S - \xi) \lambda e^{-\lambda\xi} d\xi$$

$$= S\int_0^S \lambda e^{-\lambda\xi} d\xi - \int_0^S \xi\lambda e^{-\lambda\xi} d\xi$$

$$= S(1 - e^{-\lambda S}) + Se^{-\lambda S} + \frac{e^{-\lambda S}}{\lambda} - \frac{1}{\lambda}$$

$$= \frac{\lambda S + e^{-\lambda S} - 1}{\lambda}$$

Also:

$$E(v_{n+1}|s < x_n \leq S) = \int_0^{x_n} (x_n - \xi)f(\xi)d\xi = \int_0^{x_n} (x_n - \xi)\lambda e^{-\lambda\xi}d\xi$$

$$= \frac{\lambda x_n + e^{-\lambda x_n} - 1}{\lambda}$$

Thus:

$$E(v_n\ v_{n+1}) = \int_0^{\infty} x_n E(v_{n+1}|x_n)\ (x_n)dx_n$$

$$= \int_0^{s} x_n \left(\frac{\lambda S + e^{-\lambda S} - 1}{\lambda}\right) \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta}\, dx_n$$

$$+ \int_s^{S} x_n \left(\frac{\lambda x_n + e^{-\lambda x_n} - 1}{\lambda}\right) \frac{\lambda}{1 + \lambda\Delta}\, dx_n$$

$$= \frac{(\lambda S + e^{-\lambda S} - 1)e^{-\lambda s}}{1 + \lambda\Delta} \int_0^{s} x_n e^{\lambda x_n} dx_n$$

$$+ \frac{\lambda}{1 + \lambda\Delta}\int_s^{S} x_n^2 dx_n + \frac{1}{1 + \lambda\Delta}\int_s^{S} x_n e^{-\lambda x_n} dx_n$$

$$- \frac{1}{1 + \lambda\Delta}\int_s^{S} x_n dx_n$$

So that

$$E(v_n\ v_{n+1}) = \left(\frac{\lambda S + e^{-\lambda S} - 1}{1 + \lambda\Delta}\right)\left(\frac{\lambda s + e^{-\lambda s} - 1}{\lambda^2}\right)$$

$$+ \frac{\lambda}{1 + \lambda\Delta}\left(\frac{S^3 - s^3}{3}\right) + \frac{1}{1 + \lambda\Delta}\left(\frac{S^2 - s^2}{2}\right)$$

$$- \frac{1}{1 + \lambda\Delta}\left(\frac{Se^{-\lambda S}}{\lambda} + \frac{e^{-\lambda S}}{\lambda^2} - \frac{se^{-\lambda s}}{\lambda} - \frac{e^{-\lambda s}}{\lambda^2}\right)$$

We also get:

$$E(v_n) = \int_{-\infty}^{\infty} E(v_n | x_n)\phi(x_n)dx_n$$

$$= \int_0^{\infty} x_n \phi(x_n)dx_n$$

$$= \int_0^{s} x_n \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta} dx_n + \int_s^S x_n \frac{\lambda}{1 + \lambda\Delta} dx_n$$

$$= \frac{\lambda e^{-\lambda s}}{1 + \lambda\Delta} \int_0^s x_n e^{\lambda x_n} dx_n + \frac{\lambda}{1 + \lambda\Delta} \int_s^S x_n dx_n$$

$$= \frac{\lambda e^{-\lambda s}}{1 + \lambda\Delta} \frac{se^{\lambda s}}{\lambda} - \frac{e^{\lambda s}}{\lambda^2} + \frac{1}{\lambda^2} + \frac{\lambda}{1 + \lambda\Delta}\left(\frac{S^2 - s^2}{2}\right)$$

$$= \frac{\lambda s + e^{-\lambda s} - 1}{\lambda(1 + \lambda\Delta)} + \frac{\lambda}{2(1 + \lambda\Delta)}(S^2 - s^2)$$

Also,

$$E(v_n^2) = \int_{-\infty}^{\infty} E(v_n^2 | x_n)\phi(x_n)dx_n$$

$$= \int_0^{\infty} x_n^2 \phi(x_n)dx_n$$

$$= \int_0^{s} x_n^2 \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta} dx_n + \int_s^S x_n^2 \frac{\lambda}{1 + \lambda\Delta} dx_n$$

$$= \frac{\lambda e^{-\lambda s}}{1 + \lambda\Delta} \int_0^s x_n^2 e^{\lambda x_n} dx_n + \frac{\lambda}{1 + \lambda\Delta} \int_s^S x_n^2 \, dx_n$$

$$= \frac{\lambda e^{-\lambda s}}{1 + \lambda\Delta} \left(\frac{s^2 e^{\lambda s}}{\lambda} - \frac{2s e^{\lambda s}}{\lambda^2} + \frac{2e^{\lambda s}}{\lambda^3} - \frac{2}{\lambda^3} \right)$$

$$+ \frac{\lambda}{1 + \lambda\Delta} \left(\frac{S^3 - s^3}{3} \right)$$

$$= \frac{\lambda^2 s^2 - 2s\lambda + 2 - 2e^{-\lambda s}}{\lambda^2 (1 + \lambda\Delta)} + \frac{\lambda(S^3 - s^3)}{3(1 + \lambda\Delta)}$$

Thus

$$\mathrm{Var}(v_n) = E(v_n^2) - [E(v_n)]^2$$

$$= \frac{\lambda^2 s^2 - 2s\lambda + 2 + 2e^{-\lambda s}}{\lambda^2 (1 + \lambda\Delta)} + \frac{\lambda(S^3 - s^3)}{3(1 + \lambda\Delta)}$$

$$- \left[\frac{\lambda s + e^{-\lambda s} - 1}{\lambda(1 + \lambda\Delta)} + \frac{\lambda}{2(1 + \lambda\Delta)} (S^2 - s^2) \right]^2$$

From the above, we also have:

$$\mathrm{Cov}(v_n \, v_{n+1}) = E(v_n \, v_{n+1}) - [E(v_n)]^2$$

$$= \left(\frac{\lambda S + e^{-\lambda S} - 1}{1 + \lambda\Delta} \right) \left(\frac{\lambda s + e^{-\lambda s} - 1}{\lambda^2} \right)$$

$$+ \frac{\lambda}{1 + \lambda\Delta} \left(\frac{S^3 - s^3}{3} \right) - \frac{1}{1 + \lambda\Delta} \left(\frac{S^2 - s^2}{2} \right)$$

$$- \frac{1}{1+\lambda\Delta}\left(\frac{Se^{-\lambda S}}{\lambda} + \frac{e^{-\lambda S}}{\lambda^2} - \frac{se^{-\lambda s}}{\lambda} - \frac{e^{-\lambda s}}{\lambda^2}\right)$$

$$- \left[\frac{\lambda s + e^{-\lambda s} - 1}{\lambda(1+\lambda\Delta)} + \frac{\lambda}{2(1+\lambda\Delta)}\,(S^2 - s^2)\right]^2$$

Finally,

$$\rho_{v_n\, v_{n+1}} = \frac{\mathrm{Cov}(v_n\, v_{n+1})}{\mathrm{Var}(v_n)}$$

The expressions given above for $\mathrm{Cov}(v_n\, v_{n+1})$ and Var (v_n) can then be substituted in $\rho_{v_n\, v_{n+1}}$ to get an explicit solution for $\rho_{v_n\, v_{n+1}}$ in terms of λ, s, and Δ (with $S = s + \Delta$). Here too, v_n is not a Markov process so that we cannot infer the behavior of $\rho_{v_n\, v_{n+p}}$, $p = 2, 3\ldots$ from $\rho_{v_n\, v_{n+1}}$.

4. STATISTICAL PROPERTIES OF THE REORDERS v_n

We assume the same inventory model, as described above, with zero procurement lead time. Then, if v_n = reorder in n-th period, we have:

$$v_n = \begin{cases} 0\,, & \text{if} \quad x_n > s \\ S - x_n\,, & \text{if} \quad x_n \leq s \end{cases}$$

$$x_{n+1} = \begin{cases} x_n - \xi, & \text{if } x_n > s \\ S - \xi, & \text{if } x_n \leq s \end{cases}$$

$$E(w_n\, w_{n+1}) = \int_{-\infty}^{\infty} E(w_n\, w_{n+1} \mid x_n)\phi(x_n)dx_n$$

$$= \int_{-\infty}^{s} (S - x_n)E(w_{n+1} \mid x_n)\phi(x_n)dx_n$$

$$(w_{n+1} \mid x_n \leq s) = \begin{cases} 0, & \text{if } \xi \leq S - s \\ \xi, & \text{if } \xi > S - s \end{cases}$$

$$E(w_{n+1} \mid x_n \leq s) = \int_{S-s}^{\infty} \xi f(\xi)d\xi = \int_{S-s}^{\infty} \xi \lambda e^{-\lambda\xi}d\xi$$

$$= \Delta e^{-\lambda\Delta} + \frac{e^{-\lambda\Delta}}{\lambda} = e^{-\lambda\Delta}(\Delta + \frac{1}{\lambda})$$

where $f(\xi) = \lambda e^{-\lambda\xi}$ and $\Delta = S - s$.

We thus have:

$$E(w_n\, w_{n+1}) = \int_{-\infty}^{s} (S - x_n)e^{-\lambda\Delta}(\Delta + \frac{1}{\lambda})\phi(x_n)dx_n$$

$$= e^{-\lambda\Delta}(\Delta + \frac{1}{\lambda})\int_{-\infty}^{s} (S - x_n)\, \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta}\, dx_n$$

$$= e^{-\lambda\Delta}e^{-\lambda s}\int_{-\infty}^{s} (S - x_n)e^{\lambda x_n}dx_n$$

$$= e^{-\lambda\Delta} e^{-\lambda s} \left(\frac{Se^{\lambda s}}{\lambda} - \frac{se^{\lambda s}}{\lambda} + \frac{e^{\lambda s}}{\lambda^2} \right)$$

$$= \frac{e^{-\lambda\Delta}}{\lambda^2} (\lambda S - \lambda s + 1) = \frac{e^{-\lambda\Delta}}{\lambda^2} (\lambda\Delta + 1)$$

Also,

$$E(w_n) = E(w_{n+1}) = \int_{\infty}^{\infty} E(w_n \mid x_n)\phi(x_n)dx_n$$

$$= \int_{-\infty}^{\infty} (S - x_n)\phi(x_n)dx_n$$

$$= \int_{\infty}^{s} (S - x_n) \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta} dx_n$$

$$= \frac{e^{-\lambda s}}{1+\lambda\Delta} \left(Se^{\lambda s} + \frac{e^{\lambda s}}{\lambda} - se^{\lambda s} \right)$$

$$= \frac{\lambda S + 1 - \lambda s}{\lambda(1 + \lambda\Delta)} = \frac{1}{\lambda}$$

Also,

$$E(w_n^2) = \int_{\infty}^{\infty} E(w_n^2 \mid x_n)\phi(x_n)dx_n$$

$$= \int_{\infty}^{s} (S - x_n)^2 \phi(x_n)dx_n$$

$$= \int_{\infty}^{s} (S - x_n)^2 \frac{\lambda e^{-\lambda(s - x_n)}}{1 + \lambda\Delta} dx_n$$

$$= \frac{e^{-\lambda s}}{1+\lambda\Delta} \qquad (S - x_n)^2 \lambda e^{\lambda x_n} dx_n$$

$$= \frac{e^{-\lambda s}}{1+\lambda\Delta} \left[S^2 \int_{\infty}^{s} \lambda e^{\lambda x_n} dx_n - 2S \int_{\infty}^{s} x_n \lambda e^{\lambda x_n} dx_n + \int_{\infty}^{s} x_n^2 \lambda e^{\lambda x_n} dx_n \right]$$

$$= \frac{e^{-\lambda s}}{1+\lambda\Delta}\left[S^2 e^{\lambda s} - 2S\left(se^{\lambda s} - \frac{e^{\lambda s}}{\lambda}\right) + s^2 e^{\lambda s} - \frac{2se^{\lambda s}}{\lambda} + \frac{2e^{\lambda s}}{\lambda^2}\right]$$

$$= \frac{1}{1+\lambda\Delta}\left[(S - s)^2 + \frac{2}{\lambda}(S - s) + \frac{2}{\lambda^2}\right]$$

$$\frac{(\lambda\Delta+1)^2 + 1}{\lambda^2(1+\lambda\Delta)}$$

Therefore

$$\mathrm{Var}(w_n) = E(w_n^2) - \left[E(w_n)\right]^2$$

$$= \frac{(\lambda\Delta+1)^2 + 1}{\lambda^2(\lambda\Delta+1)} - \frac{1}{\lambda^2}$$

$$= \frac{(\lambda\Delta+1)^2 - (\lambda\Delta+1) + 1}{\lambda^2(\lambda\Delta+1)}$$

$$\mathrm{Cov}(w_n\, w_{n+1}) = E(w_n\, w_{n+1}) - E(w_n)E(w_{n+1})$$

$$= E(w_n\, w_{n+1}) - \left[E(w_n)\right]^2$$

$$= \frac{e^{-\lambda\Delta}}{\lambda^2}(\lambda\Delta + 1) - \frac{1}{\lambda^2}$$

$$= \frac{e^{-\lambda\Delta}(\lambda\Delta + 1) - 1}{\lambda^2}$$

We also note that:

$$\mathrm{Cov}(w_n\ w_{n+1}) = \frac{\frac{\lambda\Delta+1}{e^{\lambda\Delta}} - 1}{\lambda^2} \leq 0 \quad \text{since} \quad \lambda\Delta+1 \leq e^{\lambda\Delta}$$

We then obtain for $\rho_{w_n\ w_{n+1}}$:

$$\rho_{w_n\ w_{n+1}} = \frac{\mathrm{Cov}(w_n w_{n+1})}{\mathrm{Var}(w_n)}$$

$$= \frac{e^{-\lambda\Delta}(\lambda\Delta+1) - 1}{\lambda^2} \quad \frac{\lambda^2(\lambda\Delta+1)}{(\lambda\Delta+1)^2 - (\lambda\Delta+1) + 1}$$

$$= \frac{(\lambda\Delta+1)\left[e^{-\lambda\Delta}(\lambda\Delta+1) - 1\right]}{(\lambda\Delta+1)^2 - (\lambda\Delta+1) + 1}$$

Also, since $\mathrm{Cov}(w_n\ w_{n+1}) \leq 0$, we note that $\rho_{w_n w_{n+1}} \leq o$, and further, since w_n is not a Markov process, we cannot infer $\rho_{w_n w_{n+p}}$, p = 2, 3, ... from $\rho_{w_n w_{n+1}}$.

II. INVENTORY MODEL WITH NON-ZERO PROCUREMENT LEAD TIME

We now revise the inventory model being considered, and consider that reorders are delivered after a specified procurement lead time. This is a more complex model, so that we have not been able to obtain as many results for the non-zero procurement lead time case as for the zero procurement lead time case. We first analyze the on-hand plus on-order stock for a general procurement lead time, and then consider the covariance and correlation of the on-hand stock level in two successive periods for the special case of a two-period procurement lead time.

1. STATISTICAL PROPERTIES OF ON-HAND PLUS ON-ORDER STOCK LEVEL z_n

We consider an (S,s) policy for this inventory model such that if z_n = sum of on-order plus on-hand stock in nth period before ordering, we then have:

$$z_{n+1} = \begin{cases} z_n - \xi, & \text{if} \quad z_n > s \\ S - \xi, & \text{if} \quad z_n \leq s \end{cases}$$

Now, if x_n = stock on hand at end of period n

y_n = stock on order at end of period n, before an ordering decision is made in period n, we then have as transition relations for x_n and y_n

If $x_n + y_n \leq s$; $\quad x_{n+1} = x_n + y_n - \xi$

$$y_{n+1} = S - x_n - y_n$$

If $x_n + y_n > s$; $\quad x_{n+1} = x_n + y_n - \xi$

$$y_{n+1} = 0$$

We can then derive the transition relations for z_n, the sum of on-hand and on-order stock in the nth period before ordering, using the fact that $z_n = x_n + y_n$. Therefore,

If $z_n \leq s$; $\quad z_{n+1} = S - \xi$

If $z_n > s$; $\quad z_{n+1} = z_n - \xi$

However, these transition relations for z_n in this case are identical to those found for x_n in the zero procurement lead time case. Therefore, we can apply all the results for the latter case to the non-zero procurement lead time case for z_n. We note that the above results have meaning only if the procurement lead time t is equal to or greater than 2.

Thus, for the exponential distribution, $f(\xi) = \lambda e^{-\lambda\xi}$, the limiting density of z is the same as that for x:

$$\phi(z) = \begin{cases} \dfrac{\lambda}{1+\lambda\Delta}; & \text{if } s < z \leq S \\[2ex] \dfrac{\lambda e^{-\lambda(s-z)}}{1+\lambda\Delta}; & \text{if } z \leq s \end{cases}$$

where $\Delta = S - s$. Referring to the transition relation for x_{n+1} above, and extending to the limit, we obtain the following:

$$\text{If } z \leq s; \qquad x = z - \xi$$

$$\text{If } z > s; \qquad x = z - \xi$$

Thus, the limiting amount of stock on hand x, is independent of the condition on z versus s. Also, since z has the same limiting distribution and transition relations as x in the zero lead time case, we can conclude that z has the same covariance and correlation structure as that developed for x. Thus, z also represents a stationary and regular Markov process. To recapitulate the characteristics of z, parallel to those obtained for x, we have the following results:

$$E\{z_n\} = \frac{s - \frac{1}{\lambda} + \lambda\frac{S^2}{2} - \frac{\lambda s^2}{2}}{1 + \lambda(S-s)}$$

$$E\{z_n\, z_{n+1}\} = \frac{Ss - \frac{S}{\lambda} - \frac{s}{\lambda} + \frac{1}{\lambda^2} + \frac{\lambda S^3}{3} - \frac{\lambda s^3}{3} - \frac{S^2}{2} + \frac{s^2}{2}}{1 + \lambda(S - s)}$$

$$E\{z_n^2\} = \frac{\lambda}{1 + \lambda(S - s)} \left(\frac{S^3}{3} - \frac{s^3}{3}\right) + \frac{1}{1 + \lambda(S - s)} \left(s^2 - \frac{2s}{\lambda} + \frac{2}{\lambda^2}\right)$$

$$\mathrm{Cov}\{z_n\, z_{n+1}\} = \frac{\frac{(S-s)^2}{12}\left[\lambda^2(S-s)^2 - 2\lambda(S-s) - 6\right]}{\left[1 + \lambda(S-s)^2\right]}$$

$$\mathrm{Var}\, z_n = \frac{\frac{\lambda^2}{12}(S-s)^4 + \frac{\lambda}{3}(S-s)^3 + (S-s)^2 + \frac{2}{\lambda}(S-s) + \frac{1}{\lambda^2}}{\left[1 + \lambda(S-s)\right]^2}$$

$$\rho_{z_n\, z_{n+1}} = \frac{\frac{(S-s)^2}{12}\left[\lambda^2(S-s)^2 - 2\lambda(S-s) - 6\right]}{\frac{\lambda^2}{12}(S-s)^4 + \frac{\lambda}{3}(S-s)^3 + (S-s)^2 + \frac{2}{\lambda}(S-s) + \frac{1}{\lambda^2}}$$

$$\rho_{z_n\, z_{n+p}} = \rho^p_{z_n\, z_{n+1}}$$

2. STATISTICAL PROPERTIES OF ON-HAND STOCK LEVEL x_n FOR INVENTORY MODEL WITH TWO-PERIOD PROCUREMENT LEAD TIME

If z_n = sum of on-hand plus on-order stock level at end of period n, before ordering.

x_n = on-hand stock level at end of period n (which can assume any real number value),

t = procurement lead time, measured in number of time periods from order to delivery,

then the following relation holds:

$$x_{n+t-1} = z_n - \xi_n - \dots - \xi_{n+t-2}$$

where ξ_n = demand in nth period.

We now specialize this relation to the case of $t = 2$, and we obtain: $x_{n+1} = z_n - \xi_n$. We will now compute $\mathrm{Cov}(x_n\, x_{n+1})$, where $\mathrm{Cov}(x_n\, x_{n+1}) = E(x_n\, x_{n+1}) - E(x_n)\, E(x_{n+1})$. We know that:

$$x_{n+1} = z_n - \xi_n$$

$$x_n = z_{n-1} - \xi_{n-1}$$

where x_n is independent of z_n versus s. Forming the product $x_n\, x_{n-1}$, and taking expectations, we get:

$$E(x_n\, x_{n+1}) = E(z_n - \xi_n)(z_{n-1} - \xi_{n-1})$$

$$= E(z_n\, z_{n-1}) - E(z_n\, \xi_{n-1}) - E(\xi_n\, z_{n-1}) + E(\xi_n\, \xi_{n-1})$$

Now, $F'{}_n\, z_{n-1})$ is given above; $E(\xi_n\, z_{n-1}) = \frac{1}{\lambda} E(z_{n-1})$ since ξ_n is independent of z_{n-1}, and $E(\xi_n) = \frac{1}{\lambda}$; and $E(\xi_n\, \xi_{n-1}) = \frac{1}{\lambda^2}$. The relation that is still to be derived is that of $E(\xi_{n-1}\, z_n)$, where z_n depends on ξ_{n-1}. We derive this relation as follows:

$$z_n = \begin{cases} S - \xi_{n-1}; & \text{if } z_{n-1} \le s \\ z_{n-1} - \xi_{n-1}; & \text{if } z_{n-1} > s \end{cases}$$

Therefore,

if $z_{n-1} \leq s$;

$$E(\xi_{n-1} z_n \mid z_{n-1}) = SE(\xi_{n-1}) - E(\xi_{n-1}^2)$$

$$= S \cdot \frac{1}{\lambda} - \frac{2}{\lambda^2}$$

If $z_{n-1} > s$;

$$E(\xi_{n-1} z_n \mid z_{n-1}) = E(\xi_{n-1} z_{n-1} \mid z_{n-1}) - E(\xi_{n-1}^2)$$

$$= z_{n-1} E(\xi_{n-1}) - E(\xi_{n-1}^2)$$

$$= \frac{z_{n-1}}{\lambda} - \frac{2}{\lambda^2}$$

Removing the condition on z_{n-1}, we obtain:

If $z_{n-1} < s$;

$$E(\xi_{n-1} z_n) = \left(\frac{S}{\lambda} - \frac{2}{\lambda^2}\right) \int_{-\infty}^{s} \frac{\lambda e^{-\lambda(s - z)}}{1 + \lambda\Delta} dz$$

$$= \left(\frac{S}{\lambda} - \frac{2}{\lambda^2}\right) \frac{e^{-\lambda s}}{1 + \lambda\Delta} \int_{-\infty}^{s} \lambda e^{\lambda z} dz$$

$$= \left(\frac{S}{\lambda} - \frac{2}{\lambda^2}\right) \frac{e^{-\lambda s}}{1 + \lambda\Delta} e^{\lambda s}$$

$$= \frac{\lambda S - 2}{\lambda^2(1 + \lambda\Delta)}$$

If $s < z_{n-1} \leq S$;

$$E(\xi_{n-1}\, z_n) = \int_s^S \left(\frac{z}{\lambda} - \frac{2}{\lambda^2}\right) \frac{\lambda}{1 + \lambda\Delta}\, dz$$

$$= \frac{\lambda}{1 + \lambda\Delta} \left[\left(\frac{S^2}{2\lambda} - \frac{2S}{\lambda^2}\right) - \left(\frac{s^2}{2\lambda} - \frac{2s}{\lambda^2}\right)\right]$$

$$= \frac{\lambda}{1 + \lambda(S - s)} \left[\frac{S^2 - s^2}{2\lambda} - \frac{2(S - s)}{\lambda^2}\right]$$

Therefore, combining the two results for $-\infty < z \leq s$, we get:

$$E(\xi_{n-1}\, z_n) = \frac{\lambda S - 2 - 2\lambda S + 2\lambda s}{\lambda^2(1 + \lambda\Delta)} + \frac{S^2 - s^2}{2(1 + \lambda\Delta)}$$

$$= \frac{2\lambda S - 4 - 4\lambda S + 4\lambda s + \lambda^2 S^2 - \lambda^2 s^2}{2\lambda^2(1 + \lambda\Delta)}$$

$$= \frac{4\lambda s - 2\lambda S + \lambda^2 S^2 - \lambda^2 s^2 - 4}{2\lambda^2(1 + \lambda\Delta)}$$

$$= \frac{(\lambda S - 1)^2 - (\lambda s - 2)^2 - 1}{2\lambda^2(1 + \lambda\Delta)}$$

Referring back to

$$E(x_n\, x_{n+1}) = E(z_n\, z_{n-1}) - E(z_n\, \xi_{n-1}) - E(\xi_n\, z_{n-1}) + E(\xi_n\, \xi_{n-1})$$

and substituting for each of the terms on the right, we obtain the following:

$$E(x_n\, x_{n+1}) = \frac{Ss - \frac{S}{\lambda} - \frac{s}{\lambda} + \frac{1}{\lambda^2} + \frac{\lambda S^3}{3} - \frac{\lambda s^3}{3} - \frac{S^2}{2} + \frac{s^2}{2}}{1 + \lambda\Delta}$$

$$- \frac{(\lambda S - 1)^2 - (\lambda s - 2)^2 - 1}{2\lambda^2(1 + \lambda\Delta)} - \frac{1}{\lambda}\left[\frac{s - \frac{1}{\lambda} + \lambda\frac{S^2}{2} - \lambda\frac{s^2}{2}}{1 + \lambda\Delta}\right] + \frac{1}{\lambda^2}$$

$$E(x_n\, x_{n+1}) = \frac{Ss - \frac{S}{\lambda} - \frac{s}{\lambda} + \frac{1}{\lambda^2} + \frac{\lambda S^3}{3} - \frac{\lambda s^3}{3} - \frac{S^2}{2} + \frac{s^2}{2}}{1 + \lambda\Delta}$$

$$- \frac{(\lambda S - 1)^2 - (\lambda s - 2)^2 + (\lambda s - 2)^2 - (\lambda S - 1)^2}{2\lambda^2(1 + \lambda\Delta)}$$

$$= E(z_n\, z_{n+1})$$

Thus for this model the expected value of the product of stock on hand in two successive periods equals the expected value of the product of stock on hand plus due-in in two successive periods, or equivalently, the expected value of the same product for the zero lead time case, which is a very interesting result.

If we compute the covariance, we get:

$$\mathrm{Cov}(x_n\, x_{n+1}) = E(x_n\, x_{n+1}) - \left[E(x_n)\right]^2$$

$$= E(z_n\, z_{n+1}) - \left[E(z_{n-1} - \xi_{n-1})\right]^2$$

$$= E(z_n\, z_{n+1}) - \left[E(z_{n-1}) - \frac{1}{\lambda}\right]^2$$

$$= E(z_n\ z_{n+1}) - \left[E(z_{n-1})\right]^2 + \frac{2E(z_{n-1})}{\lambda} - \frac{1}{\lambda^2}$$

$$= \mathrm{Cov}(z_n\ z_{n+1}) + \frac{\frac{2s}{\lambda} - \frac{2}{\lambda^2} + S^2 - s^2}{1 + \lambda\Delta} - \frac{1}{\lambda^2}$$

$$= \mathrm{Cov}(z_n\ z_{n+1}) + \frac{2\lambda s - 2 + \lambda^2 S^2 - \lambda^2 s^2 - 1 - \lambda S + \lambda s}{\lambda^2(1 + \lambda\Delta)}$$

$$= \mathrm{Cov}(z_n\ z_{n+1}) + \frac{\lambda^2 S^2 - \lambda S - \lambda^2 s^2 + 3\lambda s - 3}{\lambda^2(1 + \lambda\Delta)}$$

Now, considering Var $\{x_n\}$, we get:

$$\mathrm{Var}(x_n) = E\left\{x_n^2\right\} - \left[E\{x_n\}\right]^2$$

$$x_n = z_{n-1} - \xi_{n-1}$$

$$x_n^2 = z_{n-1}^2 - 2z_{n-1}\,\xi_{n-1} + \xi_{n-1}^2$$

$$E(x_n^2) = E(z_{n-1}^2) - \frac{2}{\lambda} E(z_{n-1}) + \frac{2}{\lambda^2}$$

$$\left[E(x_n)\right]^2 = \left[E(z_{n-1}) - \frac{1}{\lambda}\right]^2$$

$$= \left[E(z_{n-1})\right]^2 - \frac{2}{\lambda} E(z_{n-1}) + \frac{1}{\lambda^2}$$

Therefore:

$$\mathrm{Var}(x_n) = E(z_{n-1}^2) - \left[E(z_{n-1})\right]^2 + \frac{1}{\lambda^2}$$

$$x_n = z_{n-1} - \xi_{n-1}$$

$$x_n^2 = z_{n-1}^2 - 2z_{n-1}\,\xi_{n-1} + \xi_{n-1}^2$$

$$E(x_n^2) = E(z_{n-1}^2) - \frac{2}{\lambda}\,E(z_{n-1}) + \frac{2}{\lambda^2}$$

$$\left[E(x_n)\right]^2 = \left[E(z_{n-1}) - \frac{1}{\lambda}\right]^2$$

$$= \left[E(z_{n-1})\right]^2 - \frac{2}{\lambda}\,E(z_{n-1}) + \frac{1}{\lambda^2}$$

Therefore:

$$\mathrm{Var}(x_n) = E(z_{n-1}^2) - \left[E(z_{n-1})\right]^2 + \frac{1}{\lambda^2}$$

$$= \mathrm{Var}(z_n) + \frac{1}{\lambda^2}$$

Thus, we obtain

$$\rho_{x_n\,x_{n+1}} = \frac{\mathrm{Cov}(x_n\,x_{n+1})}{\mathrm{Var}(x_n)} = \frac{\mathrm{Cov}(z_n\,z_{n+1}) + \dfrac{\lambda^2 S^2 - \lambda S - \lambda^2 s^2 + 3\lambda s - 3}{\lambda^2(1 + \lambda\Delta)}}{\mathrm{Var}(z_n) + \dfrac{1}{\lambda^2}}$$